Linearised Gravitational Wave Solution to the Einstein Field Equations

A weak gravitational field has the space-time metric

$$g_{ab} = \eta_{ab} + \epsilon h_{ab} + O(\epsilon^2)$$

and its inverse is then given by

$$g^{ab} = \eta^{ab} - \epsilon h^{ab} + O(\epsilon^2)$$

where $h^{ab} = \eta^{ac}\eta^{bd}h_{cd}$, so by definition

$$\Gamma^a{}_{bc} = \frac{1}{2}\epsilon\eta^{ad}\left(h_{bd,c} + h_{cd,b} - h_{bc,d}\right) + O(\epsilon^2)$$

and it follows by definition

$$\begin{aligned} R^a{}_{bcd} &= 2\Gamma^a{}_{b[d,c]} + \underbrace{2\Gamma^a{}_{e[c}\Gamma^e{}_{d]b}}_{O(\epsilon^2)} \\ &= \frac{1}{2}\epsilon(h_{ad,bc} + h_{bc,ad} - h_{bd,ac} - h_{ac,bd}) + O(\epsilon^2). \end{aligned} \tag{1}$$

Hence

$$R_{bd} = \frac{\epsilon}{2}\left(h_d{}^a{}_{,ba} + h_{ba,}{}^a{}_d - h_{bd,}{}^a{}_a - h_a{}^a{}_{,bd}\right) + O(\epsilon^2)$$

Let $h = h^c{}_c$ and define trace-reversed $\overline{h}_{ab} = h_{ab} - \frac{1}{2}h\eta_{ab}$ as well as $\overline{h} = \overline{h}^c{}_c$, then

$$h_{ab} = \overline{h}_{ab} - \frac{1}{2}\overline{h}\eta_{ab}.$$

Since

$$-\Box\,\overline{h}_{ab} + \overline{h}_a{}^c{}_{,bc} + \overline{h}_b{}^c{}_{,ac} + \frac{1}{2}\eta_{ab}\Box\,\overline{h} = -\eta^{cd}h_{ab,cd} + \overline{h}_a{}^c{}_{,bc} + \overline{h}_b{}^c{}_{,ac} - h_{,ab}$$

using the fact that $\Gamma = O(\epsilon)$, we have

$$R_{ab} = \frac{\epsilon}{2}\left(-\Box\, \overline{h}_{ab} + \overline{h}_a{}^c{}_{,bc} + \overline{h}_b{}^c{}_{,ac} + \frac{1}{2}\eta_{ab}\,\Box\,\overline{h} \right) + O(\epsilon^2)$$

where $\Box = \eta^{ab}\nabla_a\nabla_b$.

By Einstein's field equations for a vacuum, $R_{ab} - \frac{1}{2}Rg_{ab} = 0$, whence to $O(\epsilon)$

$$-\Box\overline{h}_{ab} + \overline{h}_a{}^c{}_{,bc} + \overline{h}_b{}^c{}_{,ac} - \overline{h}^{cd}{}_{,cd}\eta_{ab} = 0. \tag{2}$$

Under the infinitesimal coordinate transformation (a gauge transformation and an isometry) $x^a \mapsto x^a + \epsilon f^a(x)$, we have

$$x^a \mapsto \tilde{x}^a - \epsilon f^a(\tilde{x}^a) + O(\epsilon^2)$$

and $\tilde{g}_{ab} = \frac{\partial x^c}{\partial \tilde{x}^a}\frac{\partial x^d}{\partial \tilde{x}^b} g_{cd}$ implies

$$\tilde{g}_{ab} = g_{ab} - 2\epsilon f_{(a,b)} + O(\epsilon^2)$$

so by comparison with the definition g_{ab} we conclude

$$h_{ab} \mapsto h_{ab} - 2f_{(a,b)} + O(\epsilon).$$

Taking this as the definition for $\tilde{h}_{ab}$ and bearing in mind that $\Gamma = O(\epsilon)$, we directly compute

$$\tilde{h}_{ad,bc} + \tilde{h}_{bc,ad} - \tilde{h}_{bd,ac} - \tilde{h}_{ac,bd} = h_{ad,bc} + h_{bc,ad} - h_{bd,ac} - h_{ac,bd}$$

as partial derivatives of f_a commute. Hence the curvature tensors are unchanged to leading order in ϵ.

Now $h \mapsto h - 2f^a{}_{,a}$ and $\overline{h}_{ab} \mapsto \overline{h}_{ab} + f^c{}_{,c}\eta_{ab} - 2f_{(a,b)}$ under the coordinate change,

$$\overline{h}^{ab}{}_{,b} \mapsto \overline{h}^{ab}{}_{,b} - f^{a,b}{}_b - f^{b,a}{}_b + f^c{}_{,cb}\eta^{ab} + O(\epsilon)$$

so by choosing $\Box f^a = \overline{h}^{ab}{}_{,b}$, then in the new coordinates $\overline{h}^{ab}{}_{,b} = 0$, from which we conclude that

$$\Box\overline{h}_{ab} = 0.$$

But this means that

$$\Box\overline{h} = 0 \Rightarrow \Box h = 0$$

and consequently

$$\Box\, h_{ab} = 0 \tag{3}$$

using the definition of $\overline{h}_{ab}$.

Consider a gravitational wave

$$h_{ab} = H_{ab} e^{ik_b x^b} \tag{4}$$

in the gauge above, where $H_{ab,c} = 0$. Then by direct substitution in (3) we find

$$k^d k_d = 0$$

and

$$H_{ab} k^b = \frac{1}{2} k_a H_b{}^b \tag{5}$$

by substitution of (4) into the gauge condition $\overline{h}^{ab}{}_{,b} = 0$.

We also note that

$$H_{ab} \mapsto H_{ab} + 2k_{(a} v_{b)} \tag{6}$$

for any v_a leaves (5) unchanged,, so solutions are arbitrary up to (6).

By direct substitution of (4) into (1) and computation, we find $R_{abcd} k^d = O(\epsilon^2)$.

Finally, if $k^a = k(1, 0, 0, 1)$ then to leading order $k_a = k(-1, 0, 0, 1)$, and (5) requires

$$\begin{aligned} H_{00} + H_{03} &= -\frac{1}{2} H \\ H_{10} + H_{13} &= 0 \\ H_{20} + H_{23} &= 0 \\ H_{30} + H_{33} &= \frac{1}{2} H \end{aligned}$$

and by (6) we can without loss of generality set $H = 0$ and $H_{0i} = 0$, leaving

$$H_{ab} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & H_{11} & H_{12} & 0 \\ 0 & H_{21} & -H_{11} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

In this we note that H_{ab} must be symmetric by symmetry of the metric tensor, and thus there are 10 degrees of freedom. Gauge condition (5) fixes 4 and (6) reduces it further by 4, leaving 2 degrees of freedom in the final form.

References

[1] *General Relativity, Examples Sheet 4.* DAMTP Mathematical Examples, Faculty of Mathematics, University of Cambridge. Retrieved on 24/05/16 at `http://www.damtp.cam.ac.uk/user/examples/D22d.pdf`.